Matching Markets and Google's Sponsored Search

Part III: Dynamics — Episode 9

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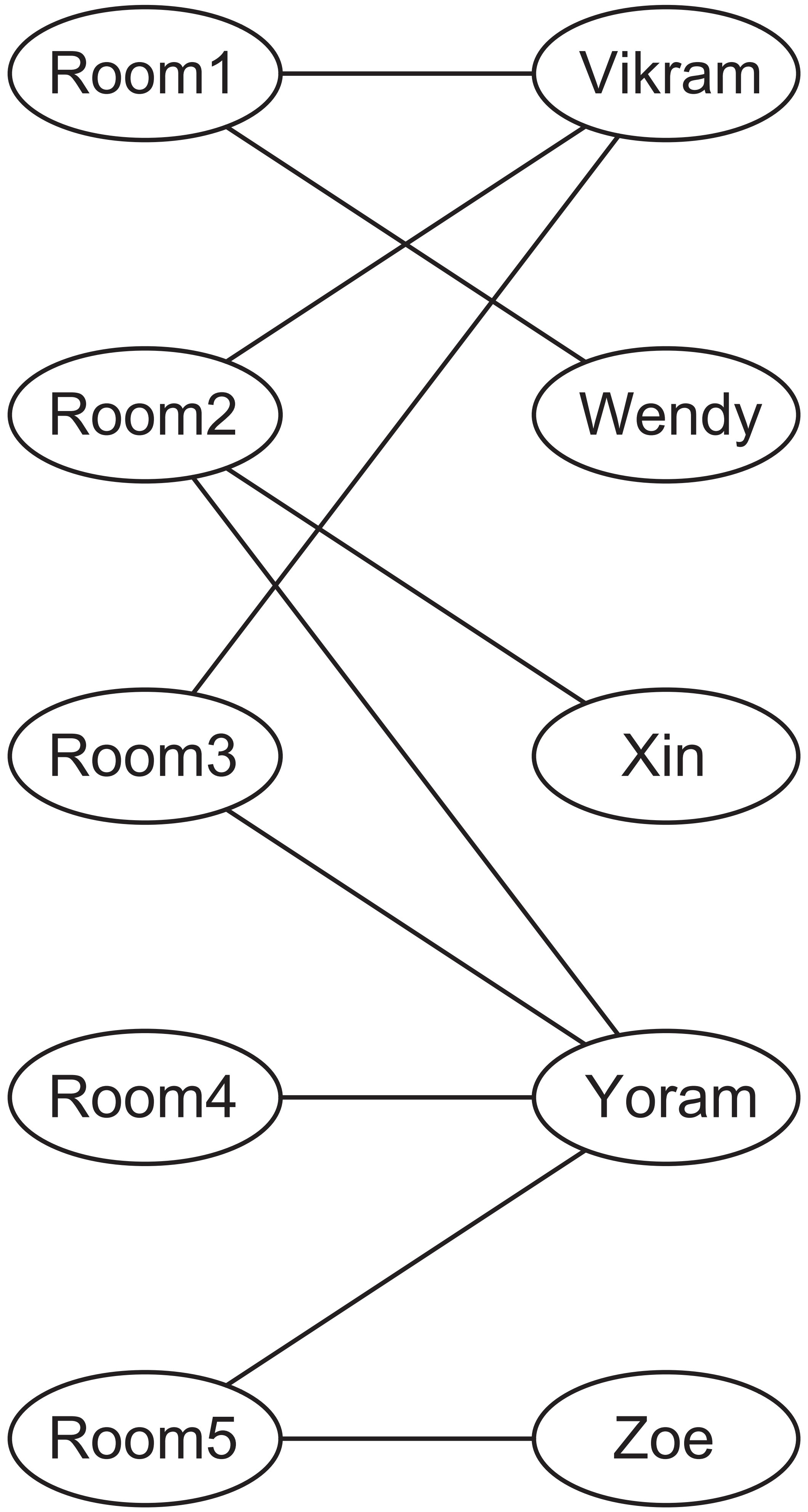
Matching Markets

(Required reading: Chapter 10.1 — 10.5)

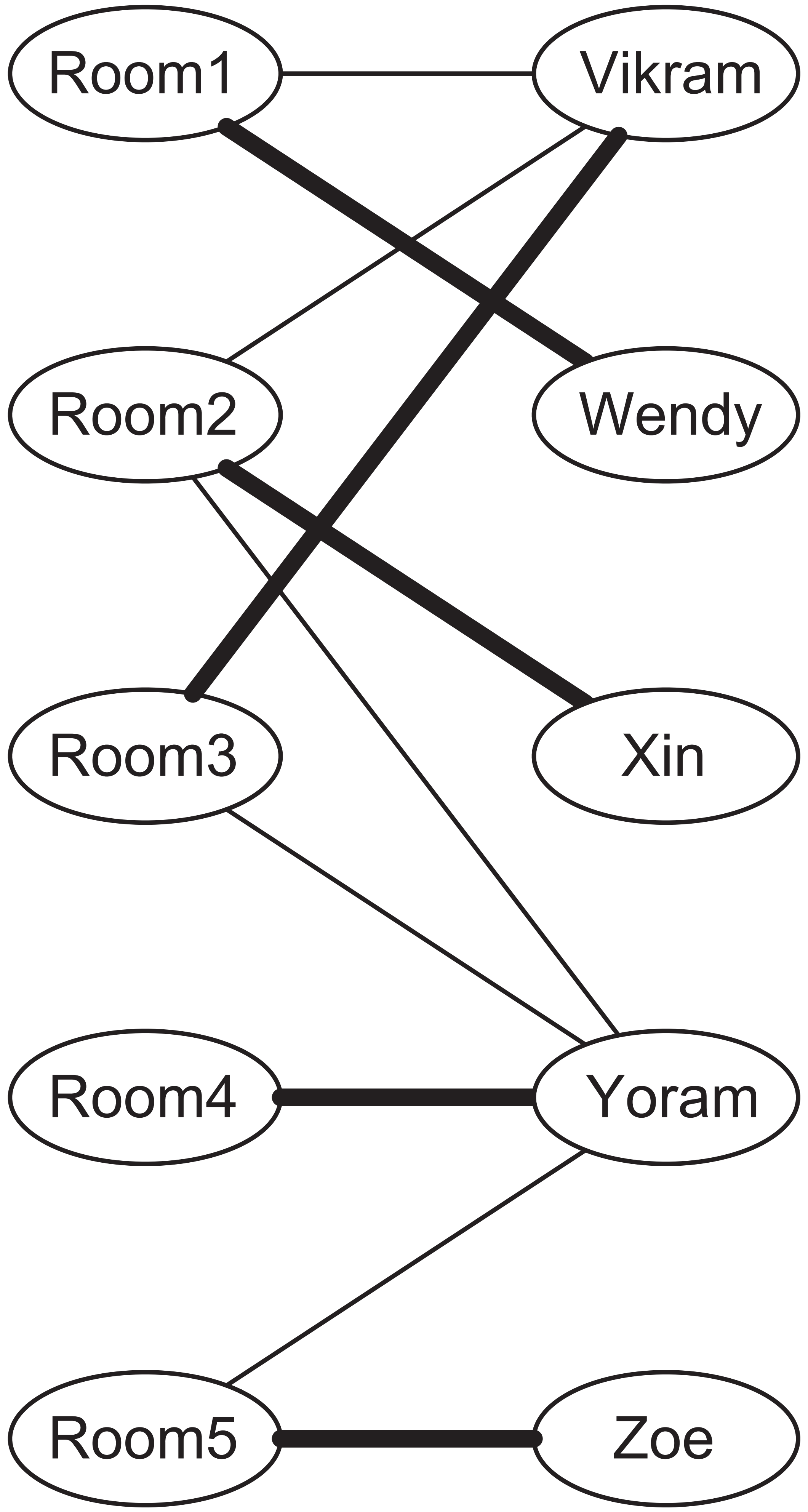
Matching Markets

- Matching markets embody a number of basic principles —
  - People naturally have different preferences for different kinds of goods
  - Prices can decentralize the allocation of goods to people
  - Such prices can in fact lead to allocations that are socially optimal
- We are going to progress through a succession of increasingly rich models

Bipartite graphs and perfect matchings



A bipartite graph with student room preferences



A perfect matching

Perfect Matching

- When there are an equal number of nodes on each side of a bipartite graph, a perfect matching is an assignment of nodes on the left to nodes on the right, in such a way that
  - each node is connected by an edge to the node it is assigned to
  - no two nodes on the left are assigned to the same node on the right
- A perfect matching can also be viewed as a choice of edges in the bipartite graph so that each node is the endpoint of exactly one of the chosen edges

What if a bipartite graph has no perfect matching? Do we need to go through all the possibilities and show that no pairing works?

A bipartite graph with no perfect matching

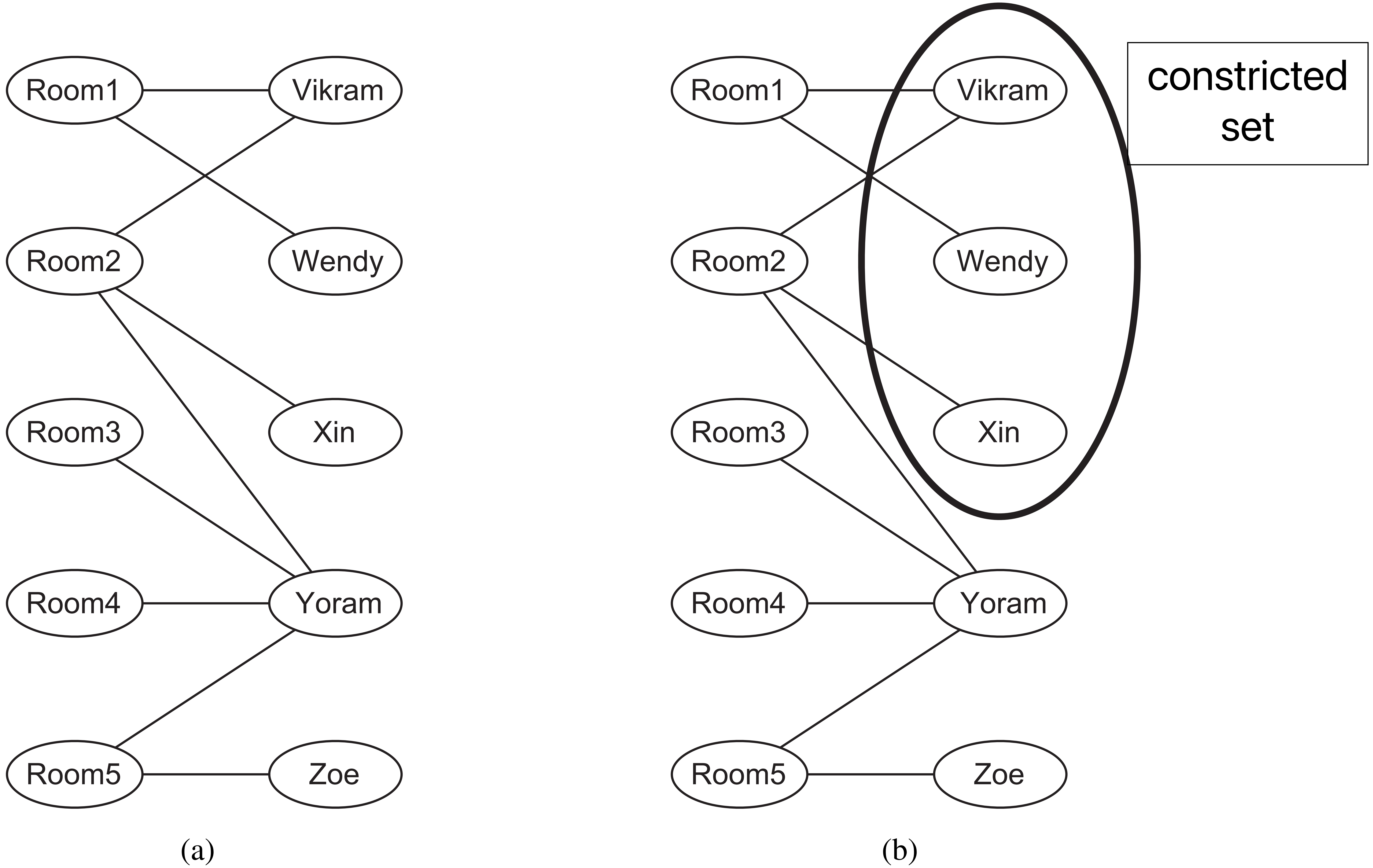


Figure 10.2. (a) A bipartite graph with no perfect matching and (b) a constricted set demonstrating there is no perfect matching.

Constricted Set and the Matching Theorem

- a set S of nodes on the right-hand side is constricted if S is strictly larger than the neighbour set of S — N(S)
  - S contains strictly more nodes than N(S) does
  - With a constricted set, there can be no perfect matching
- The Matching Theorem (1931, 1935) —

> If a bipartite graph (with equal numbers of nodes on the left and right) has no perfect matching, then it must contain a constricted set.

- This implies that a constricted set is the only obstacle to having a perfect matching!

Extending the simple model

- Rather than simple "acceptable-or-not" choices, we allow each individual to express how much they like the object, in numerical form — the "valuations"

- Optimal assignment: one that maximizes the total valuations (or the quality) of an assignment

  - Intuitively, it maximizes the total "happiness"

- We need a natural way to determine an optimal assignment

Optimal assignment: an example

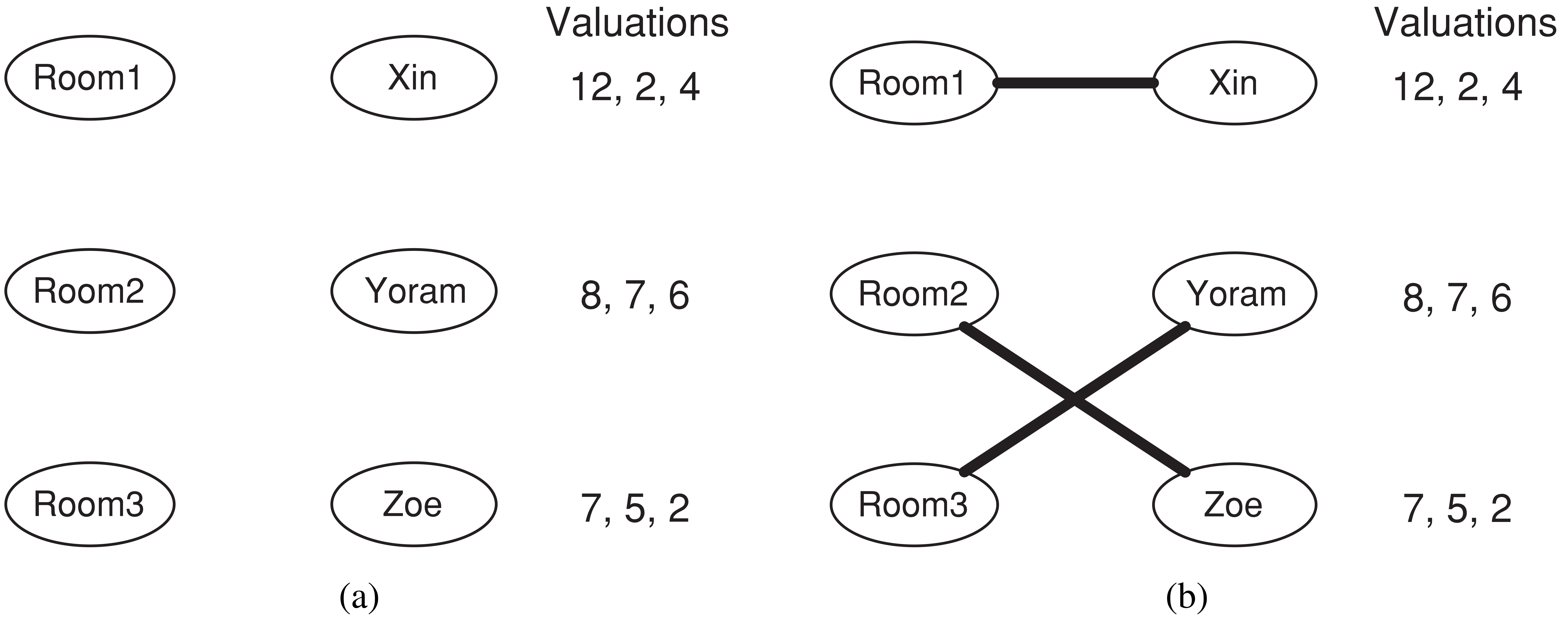


Figure 10.3. (a) A set of valuations. Each person's valuations for the objects appear as a list next to him or her. (b) An optimal assignment with respect to these valuations.

Using Prices to Decentralize the Market

- We wish to move away from a central "administrator" to determine the perfect matching or an optimal assignment
- Each individual makes her own decisions based on **prices**, in a decentralized market

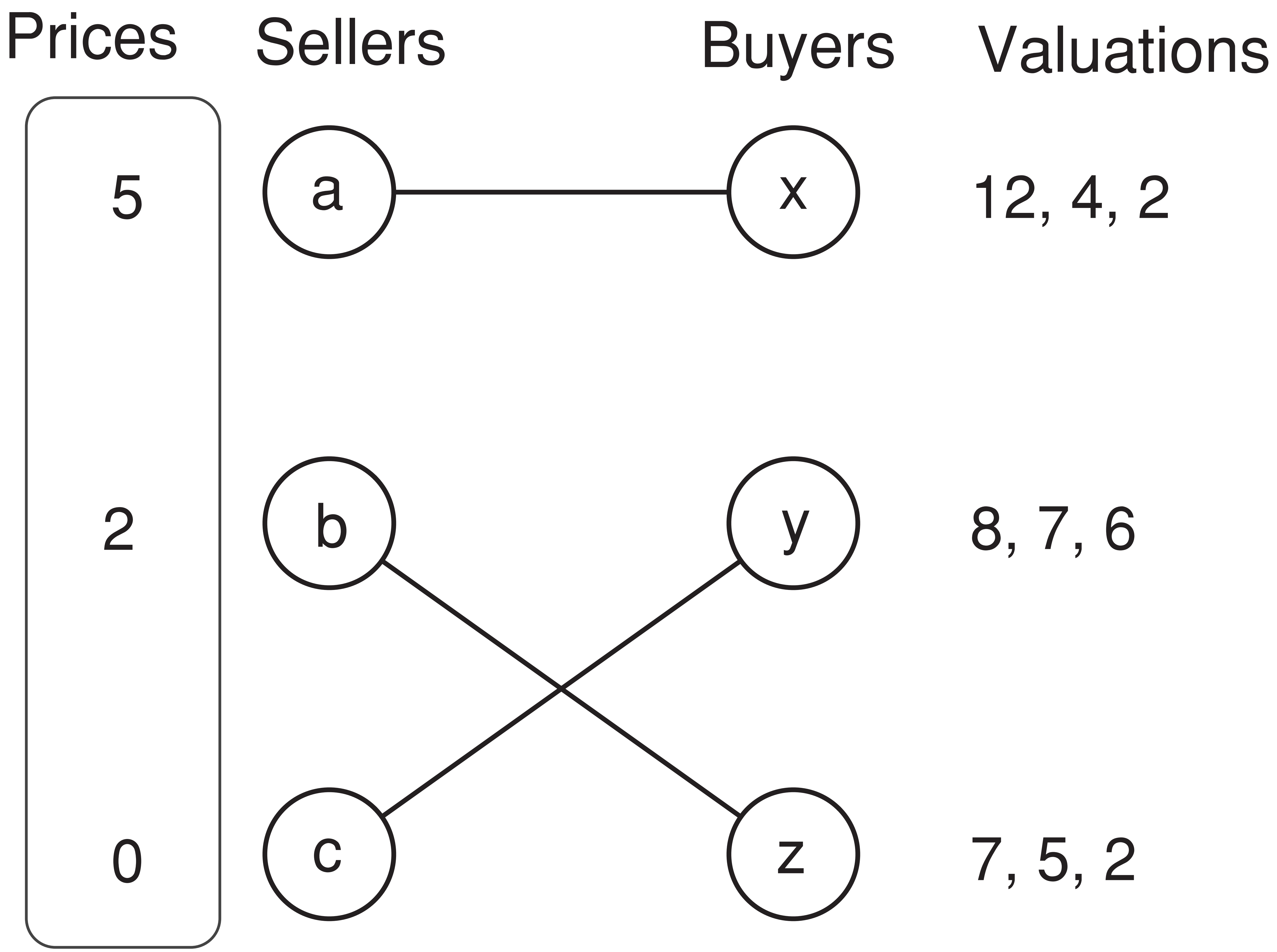
Using Prices to Decentralize the Market

- Example: the Real Estate Market
  - A collection of sellers, each having a house for sale with a price p_i
  - An equal-sized collection of buyers, each having a valuation for each house
  - The valuation that a buyer j has for the house held by seller i will be denoted v_{ij}
  - The buyer's payoff is $v_{ij} - p_i$
  - The seller(s) who maximizes a buyer's payoff is her preferred seller(s) (as long as the payoff is not negative, otherwise there's no preferred seller)

The Real Estate Market: Buyer valuations

Sellers	Buyers	Valuations
a	x	12, 4, 2
b	y	8, 7, 6
c	z	7, 5, 2

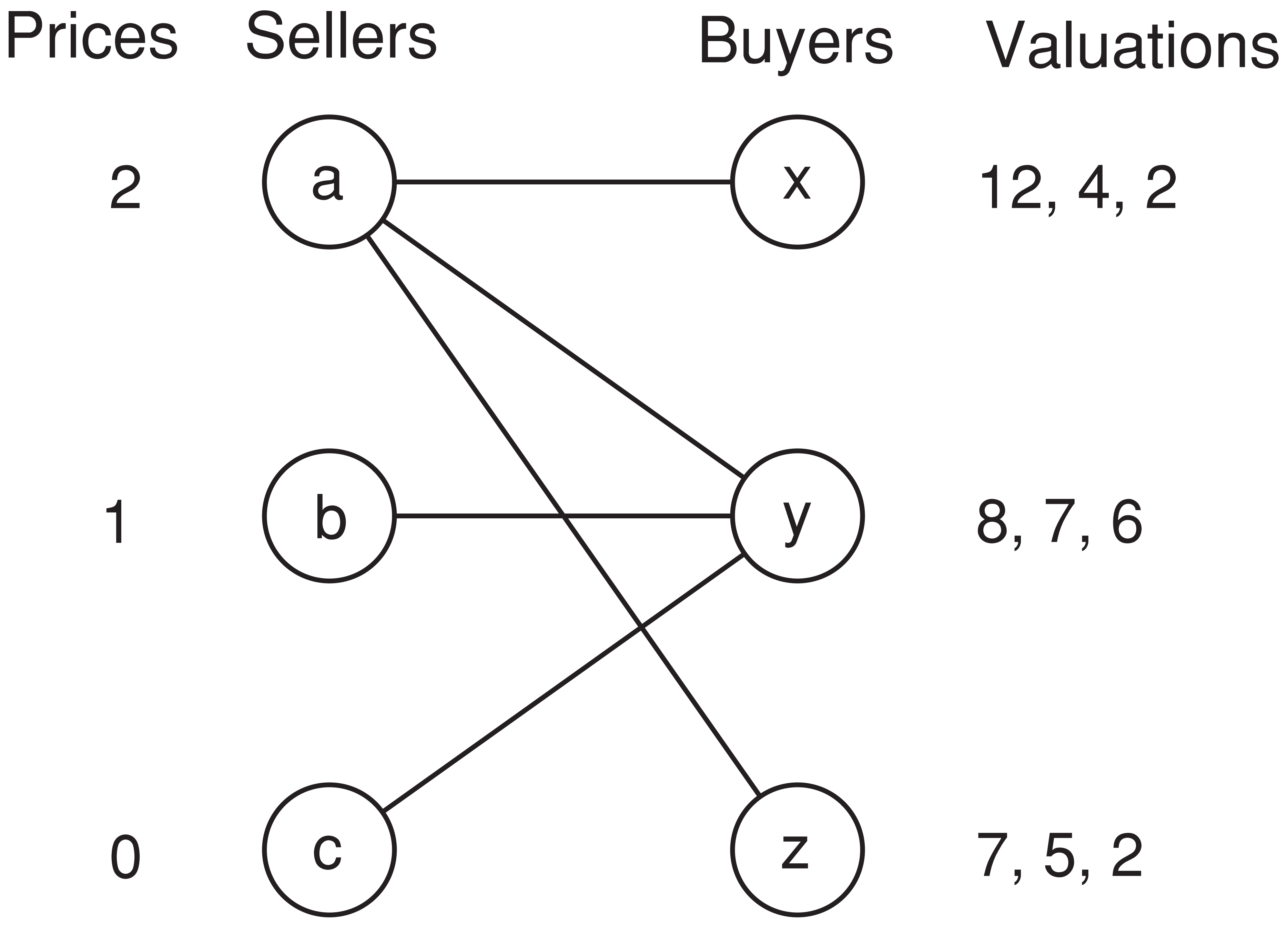
Each buyer creates a link to her preferred seller



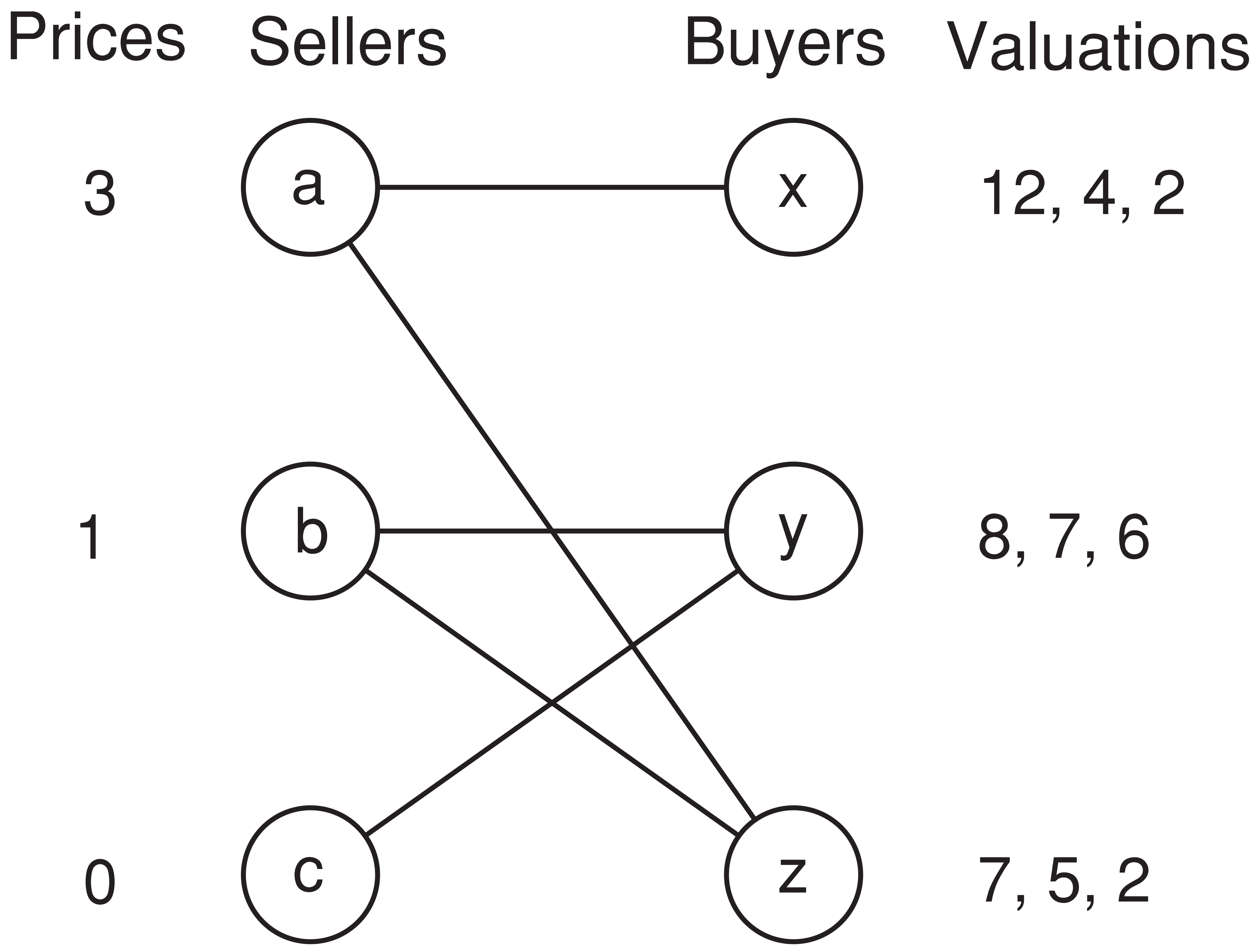
The preferred seller graph for this set of prices

Market-Clearing Prices

- The previous example shows a set of prices that is **market-clearing**, since they cause each house to get bought by a different buyer
- But not all sets of prices are market-clearing!



not market-clearing



market-clearing

A set of prices is market clearing if the resulting preferred-seller graph has a perfect matching.

Market-clearing prices: Too good to be true?

- If sellers set prices the right way, then self-interest runs its course and all the buyers get out of each other's way and claim different houses
- We've seen that such prices can be achieved in our small example; but in fact, something much more general is true!
- The existence of Market-Clearing Prices: For any set of buyer valuations, there exists a set of market-clearing prices.

Market-clearing prices and social welfare

- Just because market-clearing prices resolve the contention among buyers, causing them to get different houses, does this mean that the total valuation of the resulting assignment will be good?

- It turns out that market-clearing prices for this buyer-seller matching problem always provide **socially optimal** outcomes!

- **The optimality of Market-Clearing Prices**: For any set of market-clearing prices, a perfect matching in the resulting preferred-seller graph has the maximum **total valuation** of any assignment of sellers to buyers.

Optimality of Market-Clearing Prices

- Consider a set of market-clearing prices, and let M be a perfect matching in the preferred-seller graph
- Consider the **total payoff** of this matching, defined as the sum of each buyer's payoff for what she gets
- Since each buyer is grabbing a house that maximizes her payoff individually, M has the maximum total payoff of any assignment of houses to buyers

$$\text{Total Payoff of } M = \text{Total Valuation of } M - \text{Sum of all prices}$$

- But the sum of all prices is something that doesn't depend on which matching we choose
- So the matching M maximizes the total valuation

Alternatively, consider the total payoffs

- Consider the total payoffs of sellers and buyers
- Equivalently, we have —
- **Optimality of Market-Clearing Prices**: A set of market-clearing prices, and a perfect matching in the resulting preferred-seller graph, produces the maximum **possible sum of payoffs** to all sellers and buyers.

Why do market-clearing prices always exist?

We prove this by designing a construction algorithm that, taking an arbitrary set of buyer valuations, arrives at market-clearing prices.

Constructing a set of market-clearing prices

- The algorithm looks like an auction for multiple items to sell—
  - Initially, all sellers set their prices to 0
  - Buyers react by choosing their preferred sellers, forming a graph
  - If this preferred-seller graph has a perfect matching, we are done
  - Otherwise, there is a constricted set based on the Matching Theorem, where many buyers are interested in a smaller number of sellers
  - The sellers in the constricted set raise their price by 1
  - **Reduction**: reduce the lowest price to 0, if it is not already
  - Begin the next round of auction

Example of the construction algorithm

(a) (b)

Why must this algorithm terminate?

- Define the potential of a buyer to be the maximum payoff she can currently get from any seller
  - She will get this payoff if the prices are market-clearing
- Define the potential of a seller to be the current price he is charging
  - He will actually get this payoff if the prices are market-clearing
- Define the potential energy of the auction to be the sum of the potential of all participants, both buyers and sellers
- We are going to see that the potential energy decreases by at least one unit in each round while the auction runs

The potential energy decreases

- The potential energy is at least 0 at the start of each round
- The reduction of prices does not change the potential energy of the auction
  - If we subtract p from each price, then the potential of each seller drops by p, but the potential of each buyer goes up by p
- What happens to the potential energy of the auction when the sellers in the constricted set S all raise their prices by one unit?
  - Sellers in N(S): potential goes up by one unit in each seller
  - Buyers in S: potential goes down by one unit in each buyer
  - Since we have more buyers than sellers, the potential energy of the auction goes down by at least one unit more than it goes up

We have proved that our construction algorithm converges to a set of market-clearing prices, and that it always terminates.

Sponsored Search Markets

(Required reading: Ch. 15)

Google

jewelry

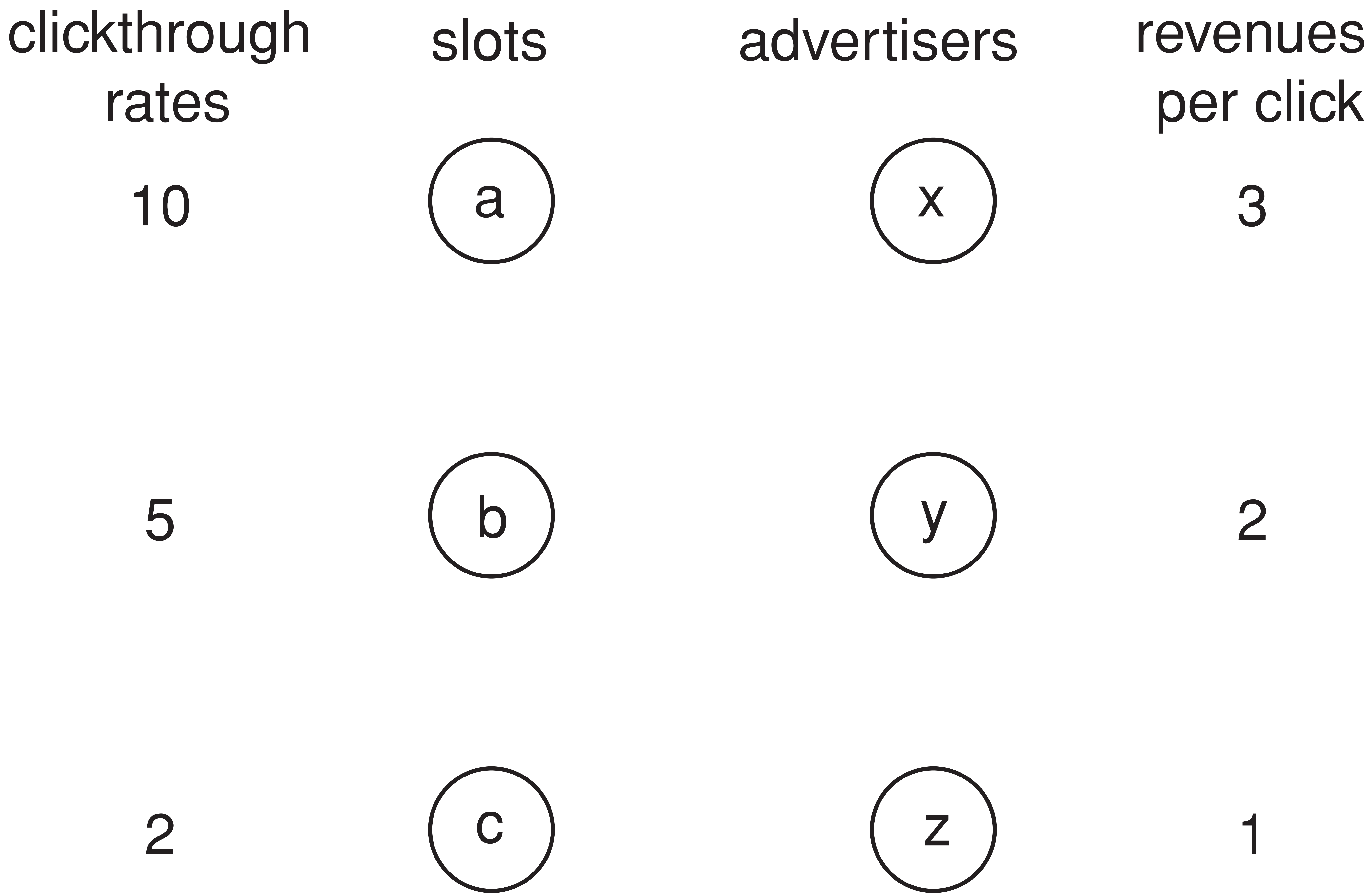
All Maps Images Shopping News More ▾ Search tools

About 1,030,000,000 results (0.55 seconds)

Clickthrough Rates and Revenues per Click

- A few assumptions before we construct a matching market between advertisers and slots
- Clickthrough rates r_i
  - Advertisers know the clickthrough rates
  - The clickthrough rate depends only on the slot, not on the ad itself
  - The clickthrough rate of a slot doesn’t depend on the ads that are in other slots
- Each advertiser has a Revenue per Click v_j
  - It is assumed to be intrinsic to the advertiser and does not depend on what’s shown on the page when the user clicked the ad

Constructing a Matching Market



Buyer's valuation: $v_{ij} = r_i v_j$

The Matching Market and Market-Clearing Prices

One problem remains

- This construction of market-clearing prices can only be carried out by Google if it actually knows the valuations of the advertisers!
- Google must rely on advertisers to report their own independent, private valuations without being able to know whether this reporting is truthful
- Google needs to encourage truthful bidding
  - Recall that truthful bidding is a dominant strategy for second-price auctions in the single-item setting
  - But we now have multiple items to sell in our market!
- Can we generalize second-price auctions to a multiple-item setting?

The Vickrey-Clarke-Groves (VCG) Principle

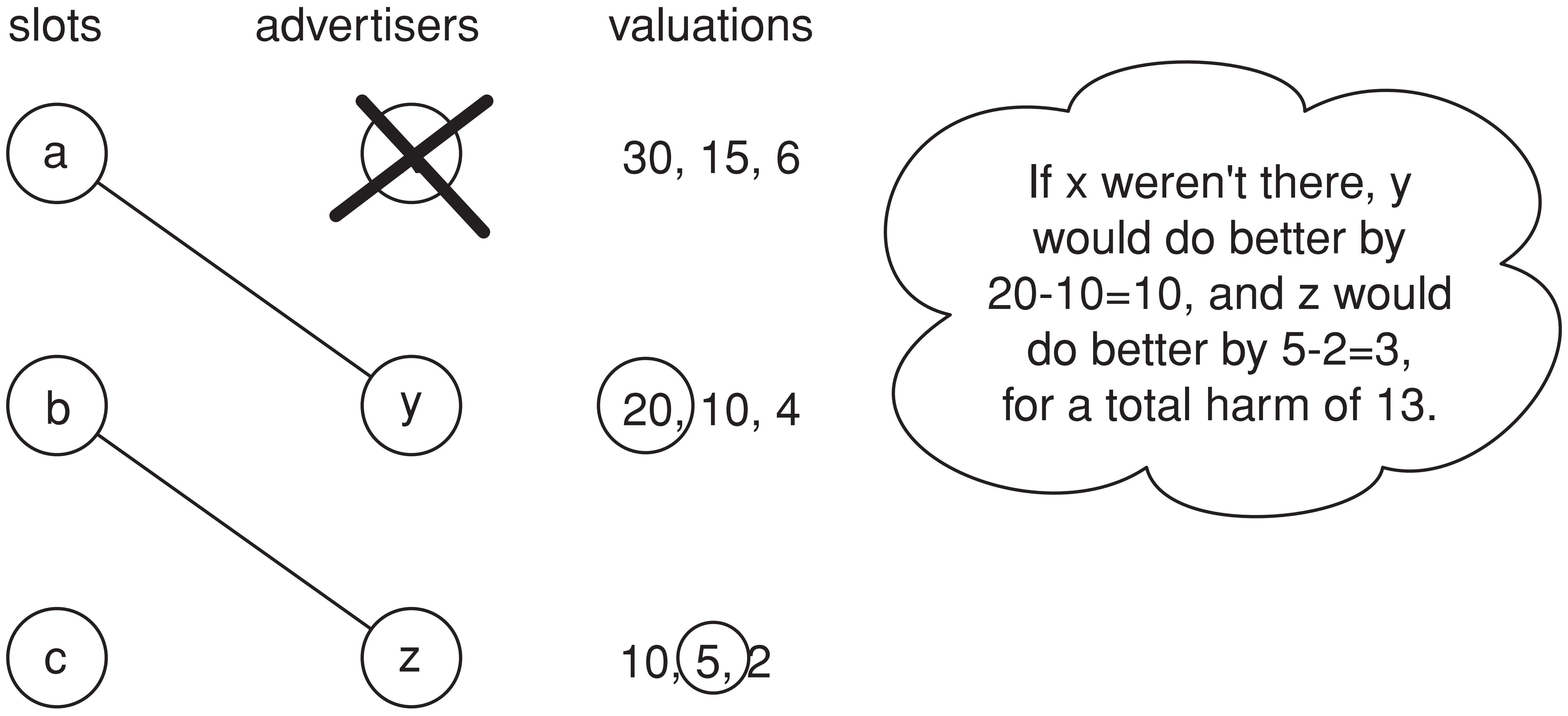
- We need to view second-price auctions in a less obvious way
- The single-item second-price auction produces an allocation that maximizes social welfare — the bidder who values the item the most gets it
- The winner of the auction is charged an amount equal to the “harm” he causes the other bidders by receiving the item
  - Suppose the bidders’ values for the item are v_1 v_2 v_3 v_4 … v_n in decreasing order
  - If bidder 1 were not present, the item would have gone to bidder 2, who values it at v_2
  - Bidders 2 through n collectively experience a harm of v_2 at the time when bidder 1 gets in!

VCG: Encouraging Truthful Reporting

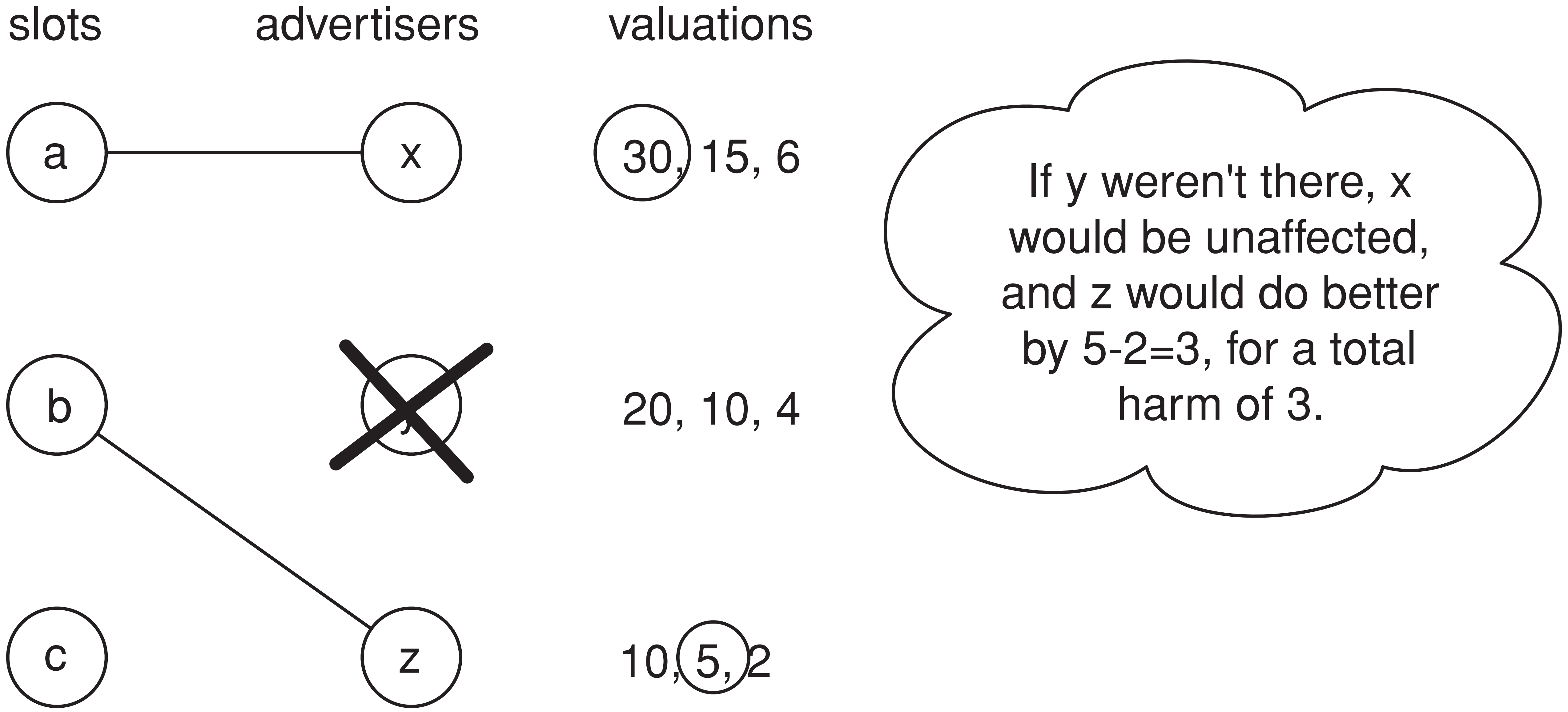
- The Vickrey-Clarke-Groves (VCG) principle (in their 1961, 1971, 1973 papers): each individual is charged a price equal to the total amount everyone would be better off if this individual weren't there
  - This is a non-obvious way to think about single-item second-price auctions
  - But it is a principle that turns out to encourage truthful reporting of private values in much more general cases!

Applying VCG to Matching Markets

- In a matching market, we have a set of buyers and a set of sellers — with equal numbers of each — and buyer j has a valuation of v_{ij} for the item being sold by seller i
- Each buyer knows her own valuations, but they are not known to other buyers or to the sellers — they have independent, private values
- We first assign items to buyers so as to maximize the total valuation
- Based on VCG, the price buyer j should pay for seller i's item is the "harm" she causes to the remaining buyers through her acquisition of this item


slots
advertisers
valuations
a
b
c
y
z
30, 15, 6
20, 10, 4
10, 5, 2
If x weren't there, y would do better by 20-10=10, and z would do better by 5-2=3, for a total harm of 13.

(a)


slots
advertisers
valuations
a
b
c
x
z
30, 15, 6
20, 10, 4
10, 5, 2
If y weren't there, x would be unaffected, and z would do better by 5-2=3, for a total harm of 3.

(b)

VCG Prices for General Matching Markets

- Let S denote the set of sellers and B denote the set of buyers
- Let V_B^S denote the maximum total valuation over all possible perfect matchings of sellers and buyers
- let S–i denote the set of sellers with seller i removed, and let B–j denote the set of buyers with buyer j removed
- Thus, the total harm caused by buyer j to the rest of the buyers is the difference between how they'd do without j present and how they do with j present —

$$p_{ij} = V_{B-j}^{S} - V_{B-j}^{S-i}$$

The VCG Price-Setting Mechanism

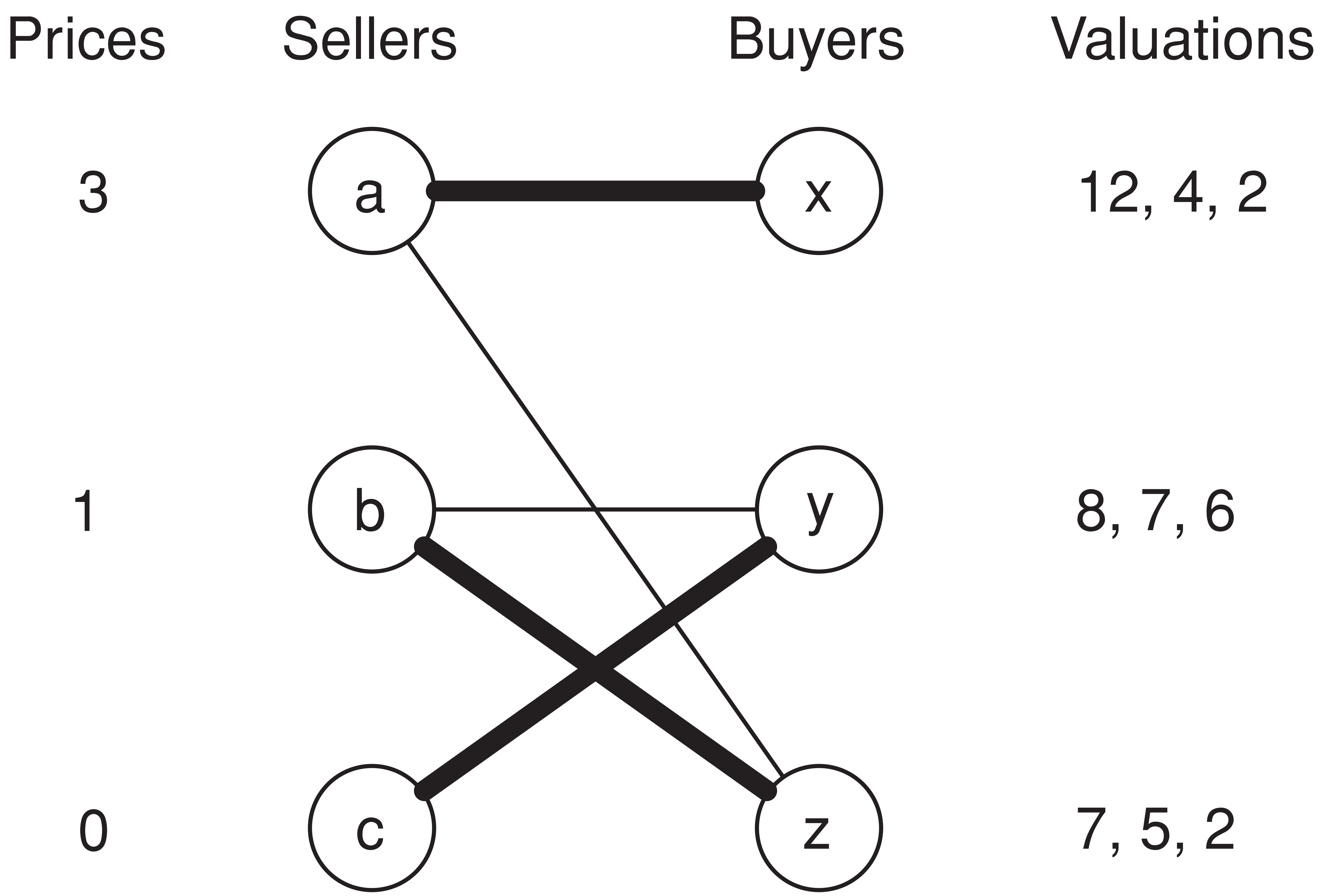
- Do the following on a price-setting authority (called "auctioneer," e.g., Google):
  - Ask buyers to announce valuations for the items (need not be truthful)
  - Choose a socially optimal assignment of items to buyers — a perfect matching that maximizes the total valuation of each buyer for what they get
  - Charge each buyer the appropriate VCG price
- What the authority did was to define a game that the buyers play —
  - They must choose a strategy (a set of valuations to announce)
  - And they receive a payoff: their valuation minus the price they pay

VCG prices vs. market-clearing prices

- The VCG prices are different from market-clearing prices
  - Market-clearing prices are posted prices, in that the seller simply announced a price and was willing to charge it to any buyer who was interested
  - VCG prices are personalized prices, they depend on both the item being sold and the buyer to whom it is being sold
  - The VCG price p_{ij} paid by buyer j for item i may be different from the VCG price p_{ik} that buyer k would pay
- The VCG prices correspond to the sealed-bid second-price auction
  - Market-clearing prices correspond to a generalization of the ascending (English) auction

Despite their definition as personalized prices, VCG prices are always market clearing.

Revisiting our example with market-clearing prices



VCG prices are always market clearing

- Suppose we were to compute the VCG prices for a given matching market
  - First determine a matching of a maximum total valuation
  - Then assign each buyer the item they receive in this matching, with a price tailored for this buyer-seller match
- Then, we go on to post the VCG prices publicly
  - Rather than requiring buyers to follow the matching used in the VCG construction, we allow any buyer to purchase any item at the indicated price!
- Despite this freedom, each buyer will in fact achieve the highest payoff by selecting the item she was assigned when VCG prices were constructed!

Being **truthful** is the **dominant strategy** in the VCG price-setting mechanism.

Claim: If items are assigned and prices computed according to the VCG mechanism, then truthfully announcing valuations is a **dominant strategy** for each buyer, and the resulting assignment **maximizes the total valuation** of any perfect matching of items and buyers.

Why is truth-telling a dominant strategy?

- Suppose that buyer j announces her valuations truthfully, and in the matching we assign her item i. Her payoff is $v_{ij} - p_{ij}$.
- If buyer j decides to lie about her valuations, either this lie does not affect the item she gets, or it does
- If she still gets the same item i, then her payoff remains exactly the same — since the price p_{ij} is computed only using announcements by buyers other than j
- If she gets a different item h, her payoff would be $v_{hj} - p_{hj}$
- We need to show there's no incentive to lie and receive item h instead of i
  - In other words, we need to show

$$v_{ij} - p_{ij} \geq v_{hj} - p_{hj}$$

or equivalently: $$v_{ij} + V_{B-j}^{S-i} \geq v_{hj} + V_{B-j}^{S-h}$$

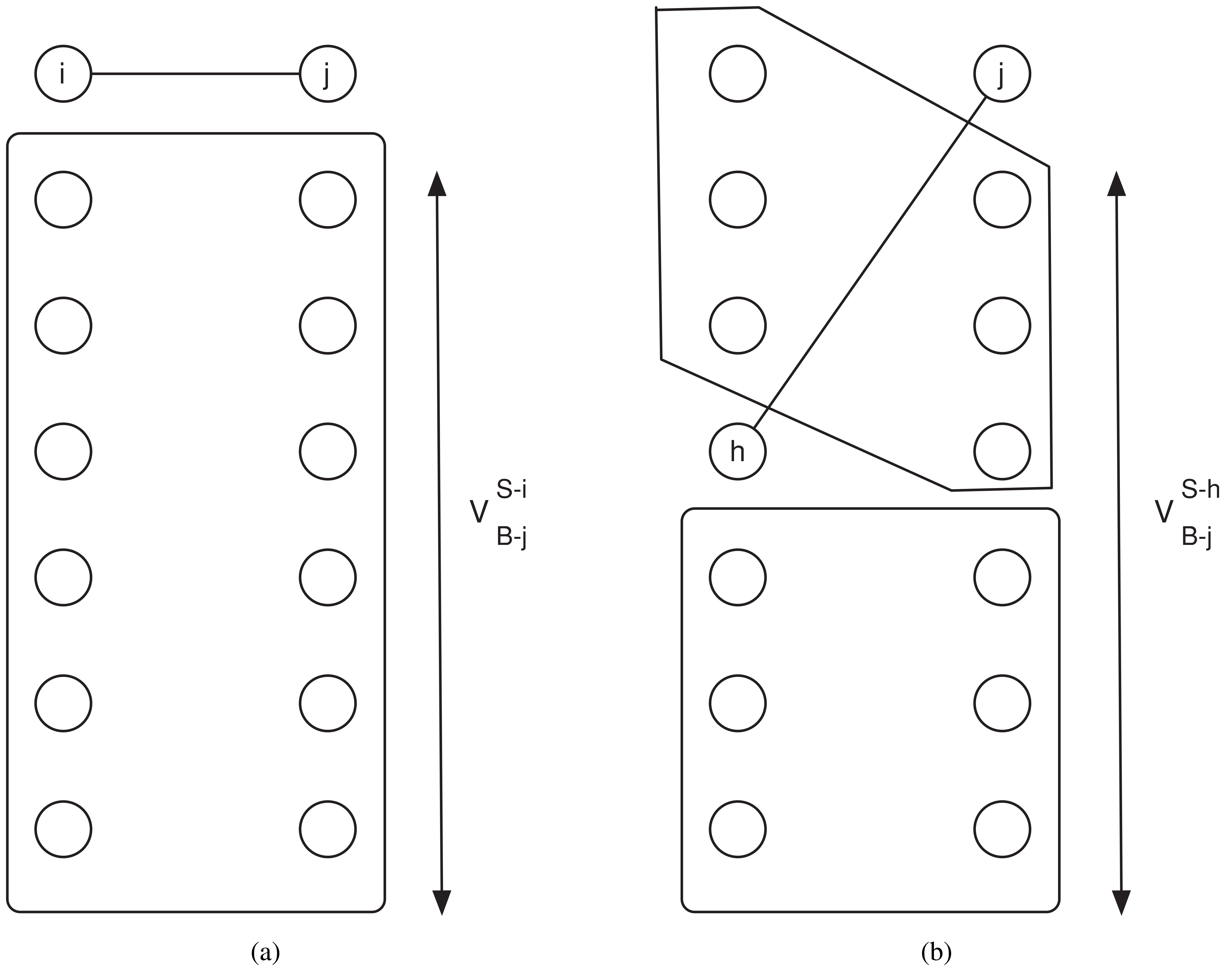


Figure 15.5. The heart of the proof that the VCG mechanism encourages truthful bidding comes down to a comparison of the value of two matchings: (a) $v_{ij} + V_{B-j}^{S-i}$ is the maximum valuation of any matching and (b) $v_{hj} + V_{B-j}^{S-h}$ is the maximum valuation only over matchings constrained to assign h to j.

Going back to keyword-based advertising

- Our discussion so far has focused on finding and achieving an assignment of advertisers to slots that maximizes the total valuation obtained by advertisers
- But of course, this is not what Google cares about!
- Instead, Google cares about its revenue: the sum of prices that it can charge for slots
  - This is easy to say, but hard to do — still a topic of research

The Generalized Second-Price Auction

- All search engines have adopted the Generalized Second-Price (GSP) auction
  - Originally developed by Google (no surprise)
  - We will see that it is a generalization of second-price auctions only in a superficial sense: it doesn't retain the nice properties of the second-price auction and VCG
- Each advertiser j announces a bid consisting of a single number b_j — the price it is willing to pay per click
- It is up to the advertiser whether or not its bid is equal to its true valuation per click, v_j
- The GSP auction awards each slot i to the ith highest bidder, at a price per click equal to (a penny higher than) the $(i+1)$st highest bid

Google AdWords Help

If the advertiser immediately below you bids US$2.00, and if that advertiser's ad is the same quality as yours (and has equal-performing extensions and ad formats), you'd typically need to bid a penny more than US$2.00 to rank higher than that advertiser and still maintain your position and ad formats. With AdWords, that's the most you'll pay (about US$2.01), whether your bid is US$3.00, US$5.00, or more.

Formulating the GSP auction as a game

- To analyze GSP, we formulate the problem as a game
  - Each advertiser is a player, its bid is its strategy, and its payoff is its valuation minus the price it pays
  - Assuming that each player knows the full set of payoffs to all players

Bad news about the GSP auction

- Truth-telling may not constitute a Nash equilibrium
- There can be multiple possible Nash equilibria
- Some of these equilibria may produce assignments of advertisers to slots that are not be socially optimal, in that they do not maximize the total advertiser valuation
- The revenue to the search engine (sum of prices) may be higher or lower than the VCG price-setting mechanism

Good news about the GSP auction

- There is always at least one Nash equilibrium set of bids for the GSP
- Among the (possibly multiple) equilibria, there is always one that does maximize total advertiser valuation

Required reading: “Networks, Crowds, and Markets,” Chapter 10.1—10.5, 15